RESONANTLY INTERACTING SPIN-1/2 FERMI GASES

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OUTLINE AND MOTIVATIONS

The context:

- System: spin-$1/2$ Fermi gas in the so-called BEC-BCS crossover. Zero-range $\uparrow - \downarrow$ interactions with s-wave scattering length a of arbitrary nonzero value ($|a|/b \to \infty$, resonant interaction).
- Realised in the lab with cold atoms and a magnetic Feshbach resonance.
- After a presentation of the basic theory tools, review some new questions raised by these systems.

Outline:

1. **Description of the system**
2. **The condensate of pairs according to BCS theory: equation of state, fermionic excitation branch, condensed fraction**
3. **A second, bosonic excitation branch: RPA and time-dependent BCS, second Josephson relation**
4. **Superfluidity: The Landau critical velocity**
5. **Temporal coherence: Thermal blurring of the condensate phase**
6. **Maximising the interaction effects: The unitary limit**

1. DESCRIPTION OF THE SYSTEM

The system:

- N fermions of mass m with two internal states $\uparrow, \downarrow$ in a trap (a cubic box of size L with periodic boundary conditions)
- Try to have a coherent gas, a fermionic counterpart of the BEC of bosons: macroscopic quantum coherence
- An attractive interaction between $\uparrow$ and $\downarrow$ atoms can lead to the formation of $\uparrow\downarrow$ pairs and to their condensation at sufficiently low temperature (BCS mechanism)
- To have a full pairing: take $N_\uparrow = N_\downarrow$
- To have as universal physics as possible: interaction of negligible range b characterised only by the s-wave scattering length a between $\uparrow$ and $\downarrow$.
- In particular, the energy of possible bound states must depend only on a, $\hbar$ and m.

- Generically, this makes the interaction in other partial waves negligible [the p-wave scattering volume for $\uparrow - \uparrow$ or $\downarrow - \downarrow$ is $O(b^3)$].
- Strong motivation: This system can be realised in the lab with cold atoms and a magnetic Feshbach resonance ($k_F b < 10^{-2}$, $|a| > 100b$) without having strong three-body losses (contrarily to p-wave resonances).

Which model interaction ?

- Negligible range: a δ interaction ?
- A three-dimensional Dirac delta

$$V(\mathbf{r}_i - \mathbf{r}_j) = g\delta(\mathbf{r}_i - \mathbf{r}_j), \quad g = \frac{4\pi\hbar^2 a}{m}$$

  is not acceptable, it has no meaning beyond the Born (first order in V) approximation.

- A Kronecker delta on a cubic spatial grid of spacing b is the nearest viable solution:

$$V(\mathrm{r}_i - \mathrm{r}_j) = \frac{g_0}{b^3}\delta_{\mathrm{r}_i,\mathrm{r}_j}$$

  with a bare coupling constant g_0 linked to the effective coupling constant g by

$$\frac{1}{g_0} = \frac{1}{g} - \int_{\mathrm{FBZ}} \frac{d^3k}{(2\pi)^3}\frac{m}{\hbar^2 k^2}$$

  with the first Brillouin zone $[-\pi/b, \pi/b[^3$, and the usual dispersion relation for the kinetic energy operator:

$$\mathrm{p}^2|\mathrm{k}\rangle = (\hbar k)^2|\mathrm{k}\rangle$$

- In the limit $b \to 0$, taken at the end of the calculations, $g_0 < 0$ so an attractive interaction.
- Pure on-site interaction so $\uparrow\downarrow$ s-wave scattering only.

- No negative-potential-collapse in the large-N limit. Only known bound state: $N=2$, $a>0$ ($E_{\text{dim}} = -\hbar^2/ma^2$).

Complements:

- Definition of the s-wave scattering length: The zero-energy two-body scattering state $\phi(\mathbf{r})$ out of the potential solves $\Delta\phi = 0$ so is of the form

$$\phi(\mathbf{r}) = A + \frac{B}{r} = A\left(1 - \frac{a}{r}\right)$$

- To obtain g_0, case of $N=2$ in the box with $\mathbf{P}=0$:

$$\langle \mathbf{k}|\phi\rangle = \frac{g_0}{L^{3/2}} \frac{\phi(\mathbf{r}=0)}{E - \hbar^2 k^2/m}$$

$$\frac{1}{g_0} = \frac{1}{L^3} \sum_{\mathbf{k}} \frac{1}{E - \hbar^2 k^2/m}$$

If $L \gg |a|$, energy shift of $\mathbf{k}=0$ is $E \sim g/L^3$, negligible as compared to $\hbar^2 k^2/m$ except for $\mathbf{k}=0$.

2. THE CONDENSATE OF PAIRS ACCORDING TO BCS THEORY: EQUATION OF STATE, FERMIONIC EXCITATION BRANCH, CONDENSED FRACTION

The BCS ground state variational Ansatz:

- Reminder: case of bosons. Pure condensate ansatz $\propto (a_\varphi^\dagger)^N|0\rangle$ leads to the Gross-Pitaevskii equation for the condensate wavefunction $\varphi(\mathbf{r})$.
- Bardeen, Cooper, Schrieffer (1957): a Glauber-type coherent state of pairs

$$|\psi_{\rm BCS}\rangle = \mathcal{N}\exp\left[b^6\sum_{\mathbf{r},\mathbf{r}'}\Gamma(\mathbf{r},\mathbf{r}')\hat{\psi}_\uparrow^\dagger(\mathbf{r})\hat{\psi}_\downarrow^\dagger(\mathbf{r}')\right]|0\rangle$$

  but now the pair creation operator is not bosonic!
- Breaks $U(1)$ symmetry but is easier to handle: Gaussian state, one can use Wick theorem (sum over all binary contractions, with permutation signs):

$$\langle\hat{b}_1\hat{b}_2\hat{b}_3\hat{b}_4\rangle = \langle\hat{b}_1\hat{b}_2\rangle\langle\hat{b}_3\hat{b}_4\rangle - \langle\hat{b}_1\hat{b}_3\rangle\langle\hat{b}_2\hat{b}_4\rangle + \langle\hat{b}_1\hat{b}_4\rangle\langle\hat{b}_2\hat{b}_3\rangle$$

- One has to minimise the grand canonical Hamiltonian:

$$H_{\mathrm{GC}} = \sum_{\mathbf{r},\sigma} b^3 \hat{\psi}_\sigma^\dagger \left(-\frac{\hbar^2}{2m} \Delta_{\mathbf{r}} \hat{\psi}_\sigma \right) + g_0 \sum_{\mathbf{r}} b^3 \hat{\psi}_\uparrow^\dagger \hat{\psi}_\downarrow^\dagger \hat{\psi}_\downarrow \hat{\psi}_\uparrow$$
$$-\mu \sum_{\mathbf{r},\sigma} b^3 \hat{\psi}_\sigma^\dagger \hat{\psi}_\sigma$$

The BCS Hamiltonian:

- One associates to H_{GC} a quadratic Hamiltonian H_{BCS} by incomplete Wick contractions:

$$\hat{b}_1\hat{b}_2\hat{b}_3\hat{b}_4 \longrightarrow \hat{b}_1\hat{b}_2\langle\hat{b}_3\hat{b}_4\rangle - \hat{b}_1\hat{b}_3\langle\hat{b}_2\hat{b}_4\rangle + \hat{b}_1\hat{b}_4\langle\hat{b}_2\hat{b}_3\rangle$$
$$+\langle\hat{b}_1\hat{b}_2\rangle\hat{b}_3\hat{b}_4 - \langle\hat{b}_1\hat{b}_3\rangle\hat{b}_2\hat{b}_4 + \langle\hat{b}_1\hat{b}_4\rangle\hat{b}_2\hat{b}_3$$
$$-[\langle\hat{b}_1\hat{b}_2\rangle\langle\hat{b}_3\hat{b}_4\rangle - \langle\hat{b}_1\hat{b}_3\rangle\langle\hat{b}_2\hat{b}_4\rangle + \langle\hat{b}_1\hat{b}_4\rangle\langle\hat{b}_2\hat{b}_3\rangle]$$

- Modifies the interaction term only. No $\uparrow - \downarrow$ coherences:

$\langle \hat{\psi}_\uparrow^\dagger \hat{\psi}_\downarrow \rangle = \mathbf{0}$. As a consequence:

$$g_0 \hat{\psi}_\uparrow^\dagger \hat{\psi}_\downarrow^\dagger \hat{\psi}_\downarrow \hat{\psi}_\uparrow \rightarrow [\hat{\psi}_\uparrow^\dagger \hat{\psi}_\downarrow^\dagger g_0 \langle \hat{\psi}_\downarrow \hat{\psi}_\uparrow \rangle + \mathbf{h.c.}]$$

$$+ [\hat{\psi}_\uparrow^\dagger \hat{\psi}_\uparrow g_0 \langle \hat{\psi}_\downarrow^\dagger \hat{\psi}_\downarrow \rangle + \uparrow \leftrightarrow \downarrow] - \text{c-number}$$

- Pairing terms involving the pairing field

$$\Delta(\mathbf{r}) \equiv g_0 \langle \hat{\psi}_\downarrow(\mathbf{r}) \hat{\psi}_\uparrow(\mathbf{r}) \rangle$$

- Hartree terms involving the densities

$$\rho_\sigma(\mathbf{r}) = \langle \hat{\psi}_\sigma^\dagger(\mathbf{r}) \hat{\psi}_\sigma(\mathbf{r}) \rangle$$

disappear in the continuous space limit $b \rightarrow 0$.

- We keep up to an additive c-number:

$$g_0 \sum_{\mathbf{r}} b^3 \hat{\psi}_\uparrow^\dagger \hat{\psi}_\downarrow^\dagger \hat{\psi}_\downarrow \hat{\psi}_\uparrow \rightarrow \sum_{\mathbf{r}} b^3 \Delta(\mathbf{r}) \hat{\psi}_\uparrow^\dagger \hat{\psi}_\downarrow^\dagger + \mathbf{h.c.}$$

Why introduce this Hamiltonian ?

- H_{BCS} and H_{GC} have the same mean value.
- For any infinitesimal variation of Γ:

$$(\delta\langle\psi_{\mathrm{BCS}}|)H_{\mathrm{GC}}|\psi_{\mathrm{BCS}}\rangle = (\delta\langle\psi_{\mathrm{BCS}}|)H_{\mathrm{BCS}}|\psi_{\mathrm{BCS}}\rangle$$

- The ground state of H_{BCS} is a BCS coherent state.
- So the ground state $|\psi_0\rangle$ of H_{BCS} is the minimiser of $\langle\psi_{\mathrm{BCS}}|H_{\mathrm{GC}}|\psi_{\mathrm{BCS}}\rangle$.
- Self-consistency conditions:

$$g_0\langle\hat{\psi}_\downarrow(\mathbf{r})\hat{\psi}_\uparrow(\mathbf{r})\rangle_0 = \Delta(\mathbf{r})$$

$$\langle\hat{\psi}^\dagger_\sigma(\mathbf{r})\hat{\psi}_\sigma(\mathbf{r})\rangle_0 = \rho_\sigma(\mathbf{r})$$

How to diagonalise H_{BCS} ?

- A quadratic Hamiltonian gives linear Heisenberg equations of motion for the fields:

$$i\hbar\partial_t \begin{pmatrix} \hat{\psi}_\uparrow \\ \hat{\psi}_\downarrow^\dagger \end{pmatrix} = \begin{pmatrix} -\frac{\hbar^2}{2m}\Delta_{\mathrm{r}} - \mu & \Delta(\mathrm{r}) \\ \Delta^*(\mathrm{r}) & -\left[-\frac{\hbar^2}{2m}\Delta_{\mathrm{r}} - \mu\right] \end{pmatrix} \begin{pmatrix} \hat{\psi}_\uparrow \\ \hat{\psi}_\downarrow^\dagger \end{pmatrix}$$

- Modal expansion:

$$\begin{pmatrix} \psi_\uparrow(\mathrm{r}) \\ \psi_\downarrow^\dagger(\mathrm{r}) \end{pmatrix} = \frac{1}{L^{3/2}} \sum_{\mathrm{k}} b_{\mathrm{k}\uparrow} \begin{pmatrix} U_k \\ V_k \end{pmatrix} e^{i\mathrm{k}\cdot\mathrm{r}} + b_{\mathrm{k}\downarrow}^\dagger \begin{pmatrix} -V_k \\ U_k \end{pmatrix} e^{-i\mathrm{k}\cdot\mathrm{r}}$$

in real, spatially homogeneous, spin-symmetric solution

$$\epsilon_{f,\mathrm{k}} \begin{pmatrix} U_k \\ V_k \end{pmatrix} = \begin{pmatrix} \frac{\hbar^2 k^2}{2m} - \mu & \Delta \\ \Delta & -\left[\frac{\hbar^2 k^2}{2m} - \mu\right] \end{pmatrix} \begin{pmatrix} U_k \\ V_k \end{pmatrix}$$

with the normalisation condition $|U_k|^2 + |V_k|^2 = 1$. This gives the BCS spectrum

$$\epsilon_{f,\mathrm{k}} = \left[\left(\frac{\hbar^2 k^2}{2m} - \mu \right)^2 + \Delta^2 \right]^{1/2}$$

and the modal amplitudes

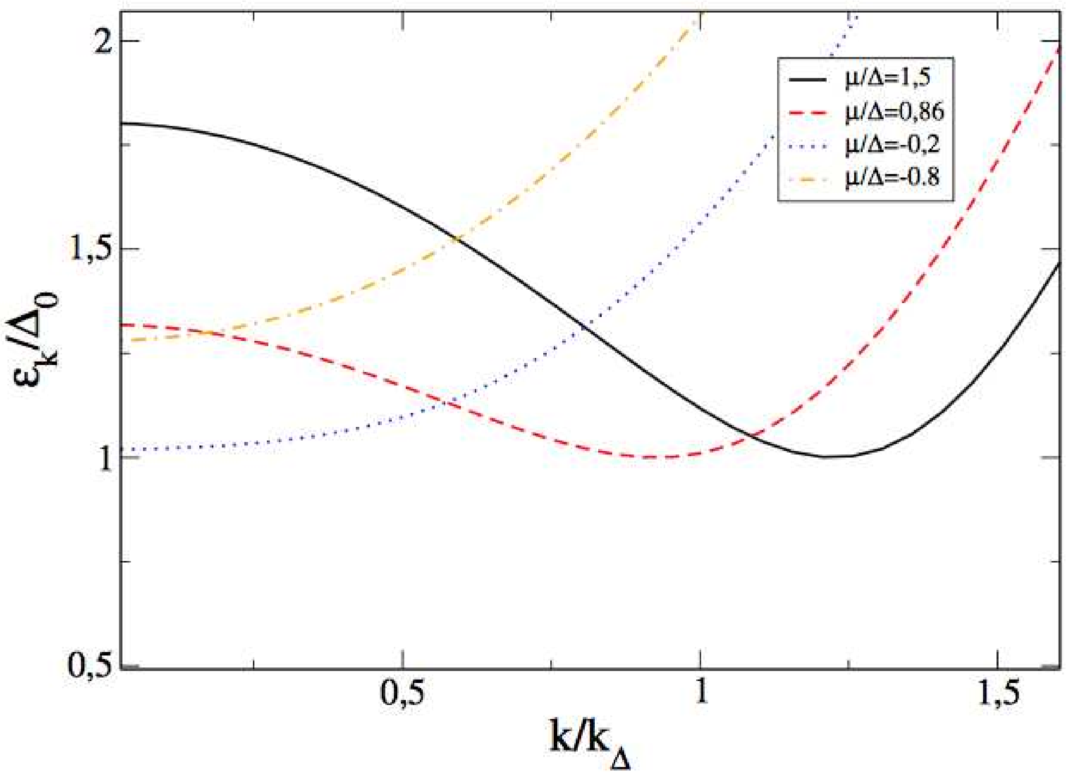
$$(U_k + iV_k)^2 = \frac{\frac{\hbar^2 k^2}{2m} - \mu + i\Delta}{\epsilon_{f,\mathrm{k}}}$$

- The operators $\hat{b}_{\mathrm{k}\sigma}$ and $\hat{b}^\dagger_{\mathrm{k}\sigma}$ obey fermionic anticommutation relations. They are annihilation and creation operators of fermionic quasiparticles. They correspond to pair-breaking excitations. Ground state=vacuum of $\hat{b}_{\mathrm{k}\sigma}$.

$$H_{\mathrm{BCS}} = \Omega_0 + \sum_{\mathrm{k},\sigma} \epsilon_{f,\mathrm{k}} \hat{b}^\dagger_{\mathrm{k}\sigma} \hat{b}_{\mathrm{k}\sigma}$$

Gap and equation of state:

- **Physical interpretation of Δ for $\mu > 0$: spectral gap = minimal pair breaking energy. Overall shape is a souvenir of the ideal Fermi sea excitation spectrum.**
- **For $\mu < 0$, minimal pair-breaking energy $= (\mu^2 + \Delta^2)^{1/2}$.**



- This was expected in the limit $k_F a \to 0^+$ ($\rho = k_F^3/3\pi^2$). A dimer exists, with a size $\ll$ mean interparticle distance. The ground state is a Bose-Einstein Condensate of dimers. BCS theory correctly predicts this to leading order: (i) the pair function $\propto$ dimer wavefunction, (ii) $\mu \sim E_{\rm dim}/2$ and $\Delta/\mu = O(k_F a)^{3/2}$.
- In the opposite BCS limit, $k_F a \to 0^-$, $\mu \to \hbar^2 k_F^2/2m$ and $\Delta/\mu \sim 8e^{-2}\exp(-\pi/2k_F|a|)$. Pairing gets fragile.
- Explicit form of the implicit equations ($E_k = \hbar^2 k^2/2m$):

$$\rho = \int \frac{d^3k}{(2\pi)^3}\left[1 - \frac{E_k - \mu}{\epsilon_{f,k}}\right] \; ; \frac{1}{g} = \int \frac{d^3k}{(2\pi)^3}\left[\frac{1}{2E_k} - \frac{1}{2\epsilon_{f,k}}\right]$$

The condensate mode φ and its pair mean number N_0:

- Generalisation to fermions of the definition of Penrose

and Onsager:

$$b^6 \sum_{\mathbf{r}_1', \mathbf{r}_2'} \rho_2(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_1', \mathbf{r}_2') \varphi(\mathbf{r}_1', \mathbf{r}_2') = N_0 \varphi(\mathbf{r}_1, \mathbf{r}_2)$$

where ρ_2 is the two-body density operator

$$\rho_2(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_1', \mathbf{r}_2') = \langle \hat{\psi}_\uparrow^\dagger(\mathbf{r}_1') \hat{\psi}_\downarrow^\dagger(\mathbf{r}_2') \hat{\psi}_\downarrow(\mathbf{r}_2) \hat{\psi}_\uparrow(\mathbf{r}_1) \rangle$$

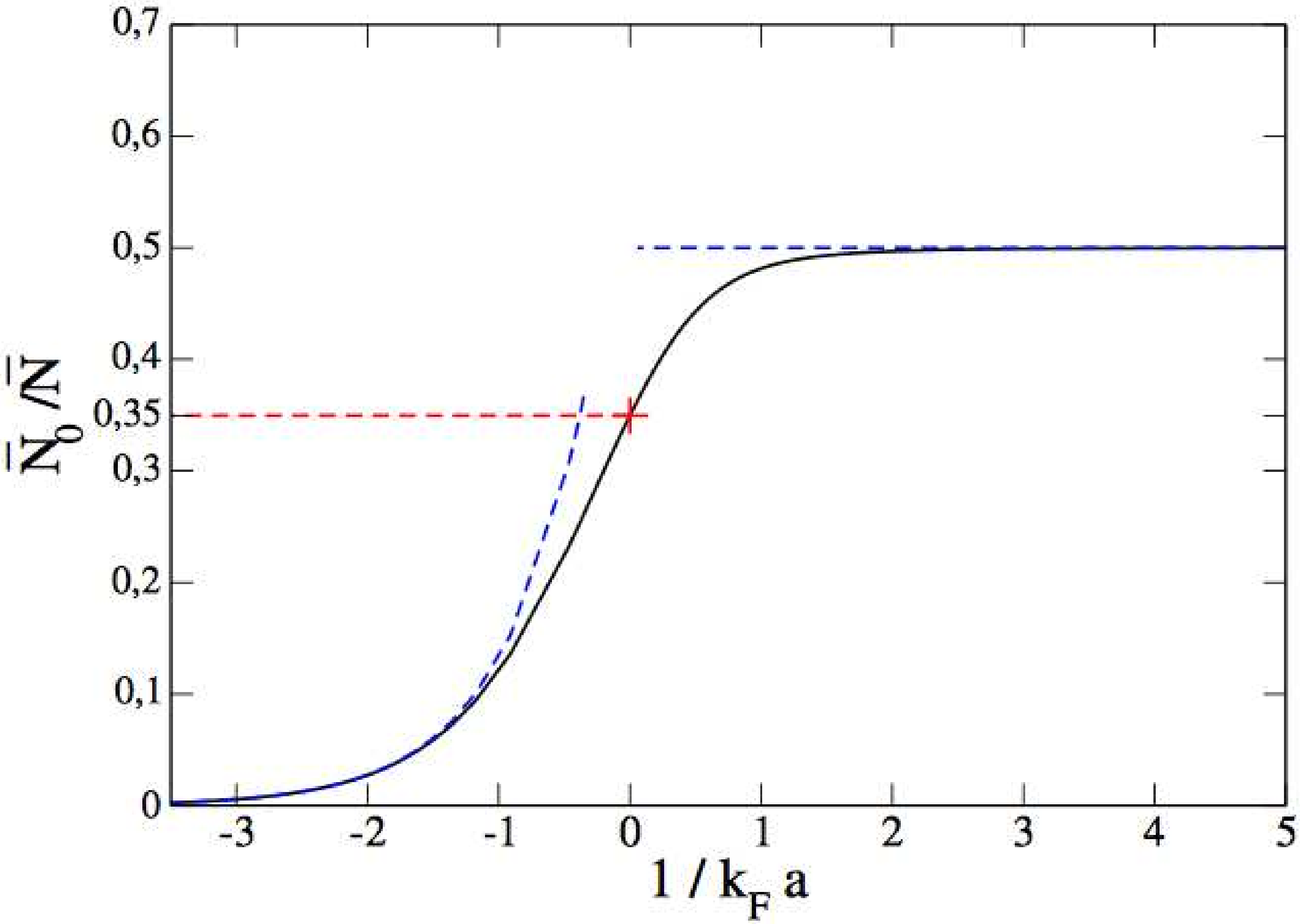
- **Only the anomalous average $\hat{\psi}_\downarrow \hat{\psi}_\uparrow$ gives a long-range contribution:**

$$N_0^{1/2} \varphi(\mathbf{r}_1, \mathbf{r}_2) = \langle \hat{\psi}_\downarrow(\mathbf{r}_2) \hat{\psi}_\uparrow(\mathbf{r}_1) \rangle = \frac{-1}{L^3} \sum_{\mathbf{k}} \frac{\Delta}{2\epsilon_{f,\mathbf{k}}} e^{i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)}$$

$$N_0 = \sum_{\mathbf{k}} \frac{\Delta^2}{4\epsilon_{f,\mathbf{k}}^2} = \mathrm{Var}\, \frac{N}{2}$$

$$\frac{N_0}{N} \underset{k_F a \to 0^-}{\sim} \frac{3\pi}{16} \frac{\Delta}{E_F} \quad \text{and} \quad \frac{N_0}{N} \underset{k_F a \to 0^+}{\longrightarrow} \frac{1}{2}$$

Number of condensed pairs over the number of fermions



A word of caution: BCS theory is only variational

- Precise measurements have been performed in cold atom systems.
- The minimum Δ of $\epsilon_{f,\mathbf{k}}$ and its location in the unitary limit [Ketterle, PRL, 2008]; Hartree shift is observed.
- Equation of state: at all accessible temperatures in the unitary limit and at zero temperature in the whole BEC-BCS crossover (Salomon, Nature and Science, 2010; Zwierlein, Science, 2012)
- The condensed fraction: Mukaiyama, Science, 2010.

3. A SECOND, BOSONIC EXCITATION BRANCH: RPA AND TIME-DEPENDENT BCS, SECOND JOSEPHSON RELATION

The BCS excitation branch is not the end of the story:

- **It is expected from hydrodynamics that any superfluid with short-range interactions has a gapless phononic excitation branch at low wavenumber q:**

$$\epsilon_{b,\mathbf{q}} \underset{q\to 0}{\sim} \hbar c q$$

with a sound velocity given by

$$mc^2 = \rho \frac{d\mu}{d\rho}$$

- **Phonons are bosons: a bosonic branch.**
- **For a pair-condensed Fermi gas, can be obtained with Anderson's RPA (1958).**

Anderson's RPA in short:

- **Take as unknows all possible operators O_2 that are bilinear in the fermionic fields**

- Write their Heisenberg equations of motion:

$$\frac{d}{dt}O_2 = \frac{1}{i\hbar}[O_2, H_{\mathrm{GC}}] = O_4$$

- Perform incomplete Wick contractions to turn O_4 into a linear superposition of the O_2's, with coefficients given by expectation values in the ground stationary BCS state.
- The eigenmodes of these linear equations give the bosonic mode dispersion relation.

Optimized implementation:

- Smarter to use the quasi-particle operators $\hat{b}_{\mathbf{k}\sigma}$ than the particle ones $\hat{a}_{\mathbf{k}\sigma}$.Use their sum and difference, and sort by total momentum change $\hbar\mathbf{q}$. Setting $\mathbf{k}_\pm = \mathbf{k} \pm \mathbf{q}/2$:

$$\hat{y}_{\mathbf{k}}^{\mathbf{q}} \text{ or } \hat{s}_{\mathbf{k}}^{\mathbf{q}} = \hat{b}_{-\mathbf{k}_+\downarrow}\hat{b}_{\mathbf{k}_-\uparrow} \mp \hat{b}^\dagger_{\mathbf{k}_+\uparrow}\hat{b}^\dagger_{-\mathbf{k}_-\downarrow}$$

$$\hat{m}_{\mathbf{k}}^{\mathbf{q}} \text{ or } \hat{h}_{\mathbf{k}}^{\mathbf{q}} = \hat{b}^\dagger_{\mathbf{k}_+\uparrow}\hat{b}_{\mathbf{k}_-\uparrow} \pm \hat{b}^\dagger_{-\mathbf{k}_-\downarrow}\hat{b}_{-\mathbf{k}_+\downarrow}$$

- **A coupling to collective variables appears:**

$$\hat{Y}^{\pm} = \frac{g_0}{L^3} \sum_{\mathbf{k}} W^{\pm}_{\mathbf{kq}} \hat{y}_{\mathbf{kq}} \text{with } W^{\pm}_{\mathbf{kq}} = U_{k_+} U_{k_-} \pm V_{k_+} V_{k_-}$$

$$\hat{y}^{\pm} = \frac{g_0}{L^3} \sum_{\mathbf{k}} w^{\pm}_{\mathbf{kq}} \hat{y}_{\mathbf{kq}} \text{with } w^{\pm}_{\mathbf{kq}} = U_{k_+} V_{k_-} \pm V_{k_+} U_{k_-}$$

- **Setting** $\epsilon^{\pm}_{\mathbf{kq}} = \epsilon_{f,\mathbf{k}_+} \pm \epsilon_{f,\mathbf{k}_-}$:

$$i\hbar \frac{d}{dt} \hat{y}^{\mathbf{q}}_{\mathbf{k}} = \epsilon^{+}_{\mathbf{kq}} \hat{s}^{\mathbf{q}}_{\mathbf{k}} + W^{-}_{\mathbf{kq}} (\hat{S}^{-} + \hat{m}^{+}) - w^{+}_{\mathbf{kq}} (\hat{M}^{-} - \hat{s}^{+})$$

$$i\hbar \frac{d}{dt} \hat{s}^{\mathbf{q}}_{\mathbf{k}} = \epsilon^{+}_{\mathbf{kq}} \hat{y}^{\mathbf{q}}_{\mathbf{k}} + W^{+}_{\mathbf{kq}} (\hat{Y}^{+} - \hat{h}^{-}) - w^{-}_{\mathbf{kq}} (\hat{y}^{-} + \hat{H}^{+})$$

$$i\hbar \frac{d}{dt} \hat{m}^{\mathbf{q}}_{\mathbf{k}} = -\epsilon^{-}_{\mathbf{kq}} \hat{h}^{\mathbf{q}}_{\mathbf{k}}$$

$$i\hbar \frac{d}{dt} \hat{h}^{\mathbf{q}}_{\mathbf{k}} = -\epsilon^{-}_{\mathbf{kq}} \hat{m}^{\mathbf{q}}_{\mathbf{k}}$$

The resulting dispersion relation:

$$I_{++}(\omega_{b,\mathrm{q}}, q) I_{--}(\omega_{b,\mathrm{q}}, q) = \hbar^2 \omega_{\mathrm{q}}^2 \left[I_{+-}(\omega_{b,\mathrm{q}}, q) \right]^2$$

$$I_{++}(\omega, q) = \int_{\mathbb{R}^3} \mathrm{d}^3 k \left[\frac{\epsilon^{+}_{\mathrm{kq}} (W^{+}_{\mathrm{kq}})^2}{(\hbar\omega)^2 - (\epsilon^{+}_{\mathrm{kq}})^2} + \frac{1}{2\epsilon_{f,\mathrm{k}}} \right]$$

$$I_{--}(\omega, q) = \int_{\mathbb{R}^3} \mathrm{d}^3 k \left[\frac{\epsilon^{+}_{\mathrm{kq}} (W^{-}_{\mathrm{kq}})^2}{(\hbar\omega)^2 - (\epsilon^{+}_{\mathrm{kq}})^2} + \frac{1}{2\epsilon_{f,\mathrm{k}}} \right]$$

$$I_{+-}(\omega, q) = \int_{\mathbb{R}^3} \mathrm{d}^3 k \frac{W^{+}_{\mathrm{kq}} W^{-}_{\mathrm{kq}}}{(\hbar\omega)^2 - (\epsilon^{+}_{\mathrm{kq}})^2}$$

- **Gives the same spectrum as other methods: (i) a Gaussian approximation of the action in a path integral framework (Strinati, 1998; Randeria, 2014), (ii) a Green's functions approach associated with a diagrammatic ap-**

proximation (Combescot, M. Kagan, Stringari, 2006).

- Has indeed a phononic start, with sound velocity given by hydrodynamic relation for BCS equation of state.
- Discussion of the branch properties will be given in section 4.

A simpler approach: time-dependent BCS

- Reminder: for weakly interacting bosons, the quantum Bogoliubov spectrum can be obtained from a linearisation of the classical field Gross-Pitaevskii equation for the condensate wavefunction $\varphi(\mathbf{r})$ around the steady state solution.
- Does the same property hold for pair-condensed fermions?
- For bosons, the fields $\varphi(\mathbf{r})$ and $\varphi^*(\mathbf{r})$ are canonically conjugate Hamiltonian variables. For fermions, one has the

same structure for the field $\mathbf{\Phi}(\mathbf{r}_1, \mathbf{r}_2)$ defined as follows (Blaizot, Ripka, 1985):

$$\underline{\underline{\Gamma}} \text{ has matrix elements } b^3\Gamma(\mathbf{r}_1, \mathbf{r}_2)$$

$$\underline{\underline{\Phi}} \text{ has matrix elements } b^3\Phi(\mathbf{r}_1, \mathbf{r}_2)$$

$$\underline{\underline{\Phi}} = -\underline{\underline{\Gamma}}(1 + \underline{\underline{\Gamma}}^\dagger \underline{\underline{\Gamma}})^{-1/2}$$

- The Gross-Pitaevskii-like equation is

$$i\hbar b^6 \partial_t \Phi(\mathbf{r}_1, \mathbf{r}_2) = \partial_{\Phi^*}\mathcal{H} \text{ with } \mathcal{H} = \langle H_{\mathrm{GC}} \rangle$$

- Linearising around the minimiser $\mathbf{\Phi_0}$,

$$i\hbar\partial_t \begin{pmatrix} \delta\Phi \\ \delta\Phi^* \end{pmatrix} = \mathcal{L} \begin{pmatrix} \delta\Phi \\ \delta\Phi^* \end{pmatrix}$$

one recovers the same excitation spectrum as the RPA.

- But the eigenvectors do not coincide. The RPA operators $\hat{m}_{\mathbf{k}}^{\mathbf{q}}$ and $\hat{h}_{\mathbf{k}}^{\mathbf{q}}$, of the form $\hat{b}^\dagger \hat{b}$, have no counterpart.

- **Why ? Their expectation value is second order in $\delta\Phi$:**

$$|\psi_{\mathrm{BCS}}\rangle = \left[1 + \sum b^6 \delta\Gamma(\mathbf{r},\mathbf{r}',t)\hat{\psi}^\dagger_\uparrow(\mathbf{r})\hat{\psi}^\dagger_\downarrow(\mathbf{r}') + O(\delta\Gamma)^2\right]|\psi^0_{\mathrm{BCS}}\rangle$$

of the form $(1+\delta\Gamma\hat{b}^\dagger\hat{b}^\dagger)|0_b\rangle$.

A spectacular consequence in the $\mathbf{q}=0$ subspace:

- **The Φ theory breaks $U(1)$ symmetry. It fixes the global phase Q to some specific value. Energy is $Q-$independent.**
- **According to Goldstone theorem, there exists an excitation branch reaching zero.**
- **Already known from Gross-Pitaevskii equation (Lewenstein, You, 1996; Castin, Dum, 1998):**

$$\mathcal{H} = \Omega_0 + \gamma P^2 + \sum \epsilon B^* B + O(\delta\Phi)^3$$

where the conserved quantity P is half the particle number and is the canonical conjugate of Q.

- Coefficient γ easy to find out:

$$\delta[E_0(N)-\mu N] \sim \frac{1}{2}E_0''(N)(\delta N)^2 = 2\frac{d\mu(N)}{dN}P^2$$

- Resulting phase evolution:

$$-\frac{\hbar}{2}\frac{dQ}{dt} = \frac{d\mu(N)}{dN}\delta N$$

- Same, more lengthy analysis for the RPA (Kurkjian, Sinatra, Castin, PRA, 2013):

$$-\frac{\hbar}{2}\frac{d\hat{Q}}{dt} = \frac{d\mu(N)}{dN}(\hat{N}-N) + \sum_{\mathbf{k},\sigma}\frac{d\epsilon_{f,\mathbf{k}}}{dN}\hat{b}^\dagger_{\mathbf{k}\sigma}\hat{b}_{\mathbf{k}\sigma}$$

  the constants of motion $\hat{m}_{\mathbf{k}}^{\mathbf{q}=0}$ and $\hat{h}_{\mathbf{k}}^{\mathbf{q}=0}$ acting as source terms.

- Interpretation: adiabatic derivative of the energy of the fermionic quasi-particles = chemical potential.

- **A quantum version of the second Josephson relation**

$$-\frac{\hbar}{2}\frac{d\theta}{dt} = \mu$$

where θ is the phase of the order parameter.

The missing piece:

- **But where is the contribution of the bosonic quasi-particles?**
- **Can be obtained by the Gross-Pitaevskii-like approach, reusing and adapting a calculation done for bosons (Sinatra, Castin, Witkowska, EPL, 2013).**
- **After quantisation through the bosonic image formalism (Blaizot, Ripka, 1985), and leaving the grand-canonical rotating frame ($\overline{\ldots}^t$ = temporal coarse-graining):**

$$-\frac{\hbar}{2}\overline{\frac{d\hat{Q}}{dt}}^t = \mu_0(\hat{N}) + \sum_{\mathbf{k},\sigma}\frac{d\epsilon_{f,\mathbf{k}}}{dN}\hat{n}_{f,\mathbf{k}\sigma} + \sum_{\mathbf{q}}\frac{d\epsilon_{b,\mathbf{q}}}{dN}\hat{n}_{b,\mathbf{q}}$$

4. SUPERFLUIDITY: THE LANDAU CRITICAL VELOCITY

'La vitesse critique de Landau d'une particule dans un superfluide de fermions", Comptes Rendus Physique 16, 241 (2015) [english version arXiv:1408.1326]

WHAT IS A CRITICAL VELOCITY ?

Defining property of a $T = 0$ superfluid: $\exists\ v_c > 0$

- an object injected in the superfluid at a velocity $v < v_c$ and coupled to it, does not experience friction and remains in motion forever
- v_c *a priori* depends on the properties of the object (its mass M), of the superfluid (its excitation spectrum $q \mapsto \epsilon_{\mathrm{q}}$) and of their interaction.
- N.B. : object prepared in its internal ground state.

Limiting case considered by Landau: fluid-object interaction $\longrightarrow 0$

- is the emission of an excitation of wavevector $\mathbf{q}$ in the superfluid compatible with conservation of momentum and unperturbed energy (Fermi golden rule) ?

- Conservation of unperturbed energy

$$\frac{1}{2}Mv^2 = \frac{1}{2}M\left(\mathrm{v} - \frac{\hbar \mathrm{q}}{M}\right)^2 + \epsilon_{\mathrm{q}} \Longleftrightarrow \hbar \mathrm{q}\cdot \mathrm{v} = \frac{\hbar^2 q^2}{2M} + \epsilon_{\mathrm{q}}$$

cannot be satisfied if $\boxed{v < v_c = \inf_{\mathrm{q}} \frac{\frac{\hbar^2 q^2}{2M} + \epsilon_{\mathrm{q}}}{\hbar q}}$

Usual criticism of the Landau critical velocity:

- Approximation (done here): include minimal nonzero number of elementary excitations. Gives a nonzero v_c.
- But it is argued that, if one includes the excitation of a large vortex annulus of radius R,

$$q \propto R^2 \text{ and } \epsilon_{\mathrm{q}} \propto R \ln R$$

one gets a vanishing $O(R \ln R/R^2)$ critical velocity for our infinite superfluid.

- Does not apply however for a finite mass object:

$$v_c^{\text{vortex}} \underset{M\to+\infty}{=} O\left(\frac{(\ln M)^{2/3}}{M^{1/3}}\right)$$

- Lychkovskiy theorem [PRA (2015)]: for a finite M and a weak enough superfluid-object nonnegative interaction potential U, there exists a nonzero critical velocity and it is almost given by Landau formula (with all possible excitations of the superfluid included):

$$|\mathrm{v}(t=0)-\mathrm{v}(t=+\infty)| \le \frac{\rho\int d^3r U(\mathrm{r})}{M[v_c - v(t=0)]}$$

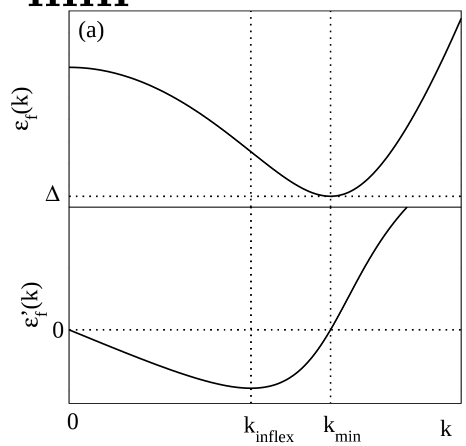
- In this lecture object = a particle (an atom). Experiment already done in a superfluid of bosons (Ketterle, PRL, 2000). Generalisation to a superfluid of fermions [infinite mass case: Ketterle, PRL, 2007)].

CONTRIBUTION $v_{c,f}$ OF THE FERMIONIC BRANCH

Pair-breaking excitation spectrum of BCS theory:

$$\epsilon_{f,\mathbf{k}} = \left[\left(\frac{\hbar^2 k^2}{2m} - \mu \right)^2 + \Delta^2 \right]^{1/2}$$

- We restrict to the fermion-like regime of a positive chemical potential $\mu > 0$ (in the boson-like regime, v_c determined by the bosonic branch)
- Gap Δ, located at $k_{\rm min} > 0$.



- A trap to avoid: fermionic excitations are created by pairs due to conservation of the number of fermions (cf. density-density superfluid-object coupling)

$$\psi_\sigma(\mathbf{r}) = \frac{1}{L^{3/2}} \sum_{\mathbf{k}} U_{k\sigma} b_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r}} + V_{k\sigma} b^\dagger_{\mathbf{k}-\sigma} e^{-i\mathbf{k}\cdot\mathbf{r}}$$

- Emission *a minima* of a two excitations of wavevectors $\mathbf{k}_1$ and $\mathbf{k}_2$ so, at fixed total wavevector $\mathbf{q}$, effective excitation branch in Landau reasoning:

$$\boxed{\epsilon^{\mathrm{eff}}_{f,\mathbf{q}} = \inf_{\mathbf{k}_1}[\epsilon_{f,\mathbf{k}_1} + \epsilon_{f,\mathbf{k}_2=\mathbf{q}-\mathbf{k}_1}]}$$

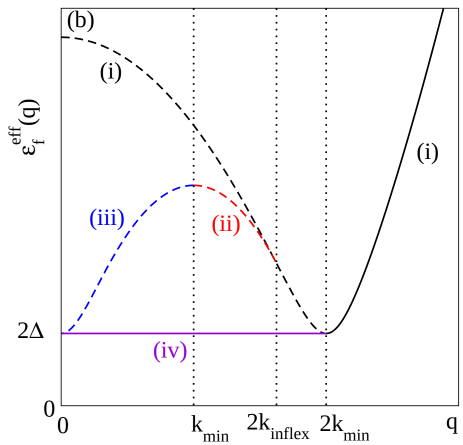
- $\epsilon_{f,\mathbf{k}} = \epsilon_f(k)$ is a smooth function of $\mathbf{k}$ that diverges at infinity, so zero gradient at minimum:

$$\epsilon'_f(k_1)\hat{\mathbf{k}}_1 = \epsilon'_f(k_2)\hat{\mathbf{k}}_2$$

- This generates four cases:

$$\mathrm{(i)}\mathbf{k}_1 = \mathbf{k}_2 = \frac{\mathbf{q}}{2}, \ \mathrm{(ii)}\hat{\mathbf{k}}_1 = \hat{\mathbf{k}}_2, k_1 \neq k_2, \ \mathrm{(iii)}\hat{\mathbf{k}}_1 = -\hat{\mathbf{k}}_2, \ \mathrm{(iv)}\epsilon'_f(k_1) = \epsilon'_f(k_2) = 0$$

- Minimisation is trivial for $q < 2k_{\rm min}$: k_1 and k_2 are located in the minimum of $\epsilon_{f,\mathrm{k}}$, $k_1 = k_2 = k_{\rm min}$.



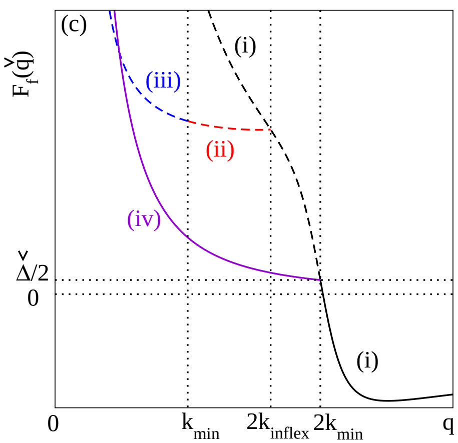
$$q < 2k_{\rm min} : \epsilon^{\rm eff}_{f,\mathrm{q}} \overset{(iv)}{=} 2\Delta$$
$$q > 2k_{\rm min} : \epsilon^{\rm eff}_{f,\mathrm{q}} \overset{(i)}{=} 2\epsilon_f(q/2)$$

Minimisation over q:

- Use μ as unit of energy, $(2m\mu)^{1/2}$ as unit of momentum, $(\mu/2m)^{1/2}$ as unit of velocity. Then $v_{c,f}$ is the minimum of $v_f(q) = \alpha q + \frac{\epsilon_f^{\rm eff}(q)}{q}$ with $\alpha = \frac{m}{M}$. Zero q-derivative:

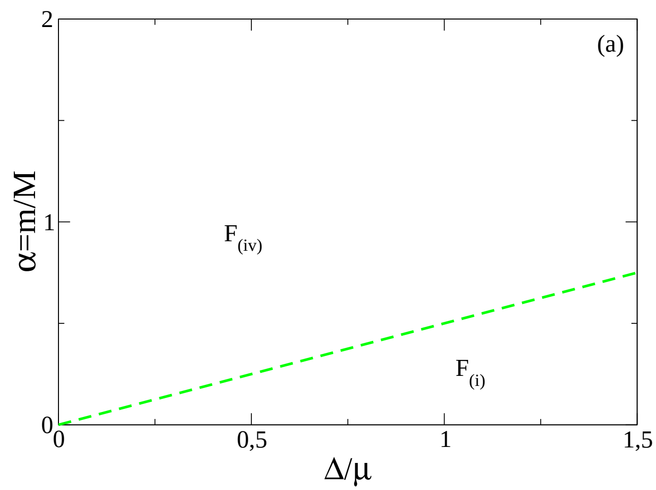
$$0 = \alpha - F_f(q_0) \ \text{ with } \ F_f(q) = -\frac{d}{dq}\frac{\epsilon_f^{\rm eff}(q)}{q}$$

- Graphical solution of $F_f(q_0) = \alpha$:



$$\boxed{\begin{array}{l} \alpha > \Delta/2 : \text{type (iv)},\ q_0 < 2k_{\min} \\ \alpha < \Delta/2 : \text{type (i)},\ q_0 > 2k_{\min} \end{array}}$$

- Across the (i)-(iv) boundary: q_0 is continuous, so is $\frac{d}{d\alpha}v_{c,f} = q_0$, but $\frac{d^2}{d\alpha^2}v_{c,f}$ is discontinuous.



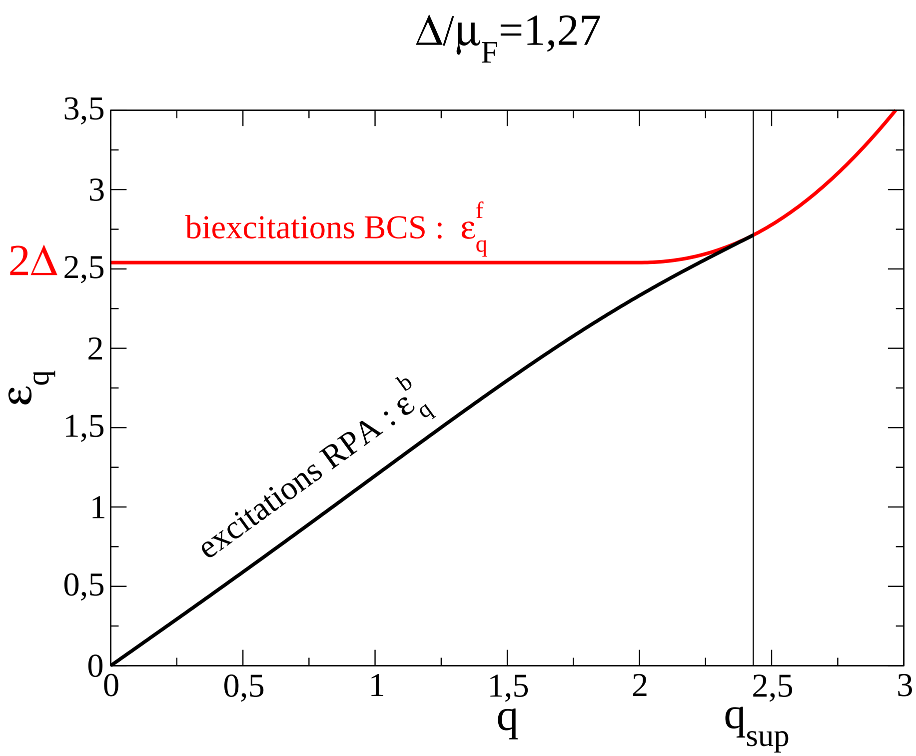
CONTRIBUTION $v_{c,b}$ OF THE BOSONIC BRANCH

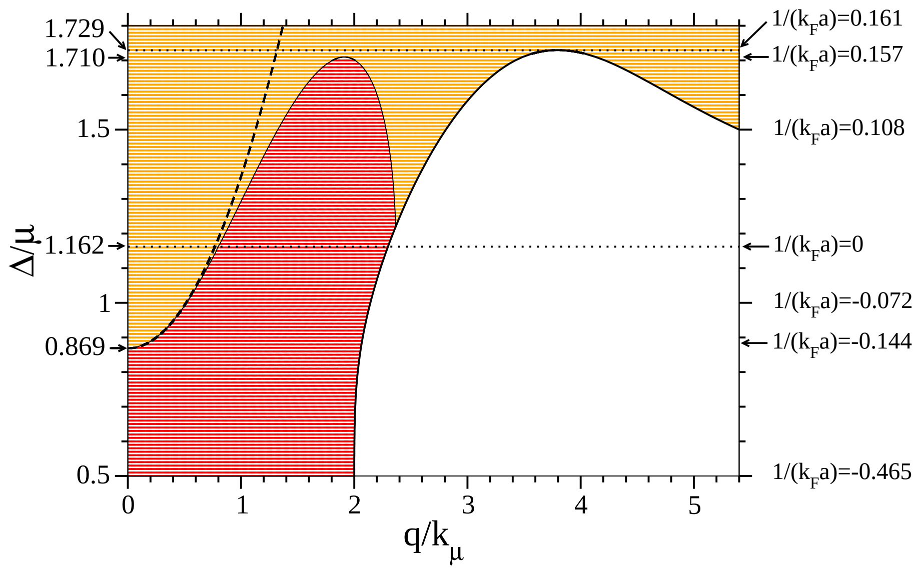
General properties of this branch:

- excitation of the pair center of mass (Anderson, 1958)
- at low q, is phononic (sound wave): $\epsilon_{b,\mathrm{q}} \sim \hbar c q$
- remains below fermionic biexcitation "roof" (would be otherwise unstable): $\epsilon_{b,\mathrm{q}} \leq \epsilon_{f,\mathrm{q}}^{\mathrm{eff}}$
- Its wavenumber existence domain can be $[0, q_{\mathrm{sup}}]$ (for $k_F a < 0$) or $[0, q_{\mathrm{sup}}] \cup [q_{\mathrm{inf}}, +\infty[$ or $[0, +\infty[$ $(1/k_F a > 0.16)$. One has $q_{\mathrm{sup}} > 2k_{\mathrm{min}}$ always.
- Reaches the biexcitation roof tangentially at q_{sup} :

$$\epsilon_b(q_{\mathrm{sup}}) = \epsilon_f^{\mathrm{eff}}(q_{\mathrm{sup}}) \text{ and } \frac{d}{dq}\epsilon_b(q_{\mathrm{sup}}) = \frac{d}{dq}\epsilon_f^{\mathrm{eff}}(q_{\mathrm{sup}})$$

- Entirely concave (convex) in the BCS (BEC) limit, rich concavity properties in between.





[taken from Kurkjian, Castin, Sinatra, PRA (2016)]

Minimisation of $v_b(q) = \alpha q + \frac{\epsilon_b(q)}{q}$:

- **We discuss here minimisation over** $[0, q_{\text{sup}}]$.
- **Three possible cases:**

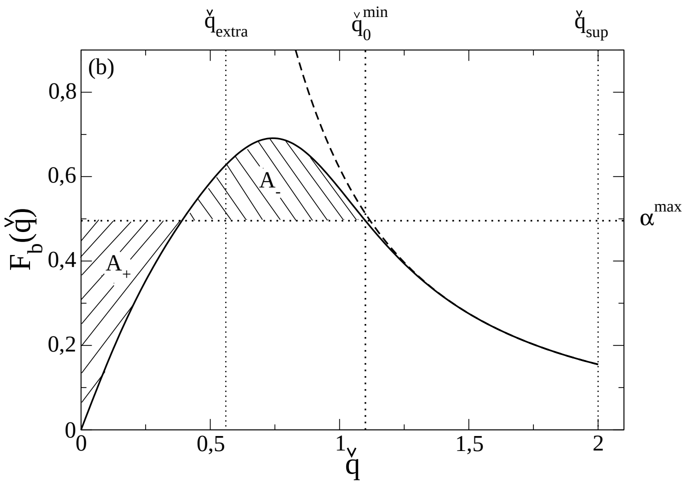
$$(0) : q_0 = 0, \quad (q_0) : 0 < q_0 < q_{\text{sup}}, \quad (q_{\text{sup}}) : q_0 = q_{\text{sup}}$$

- **Median case:**

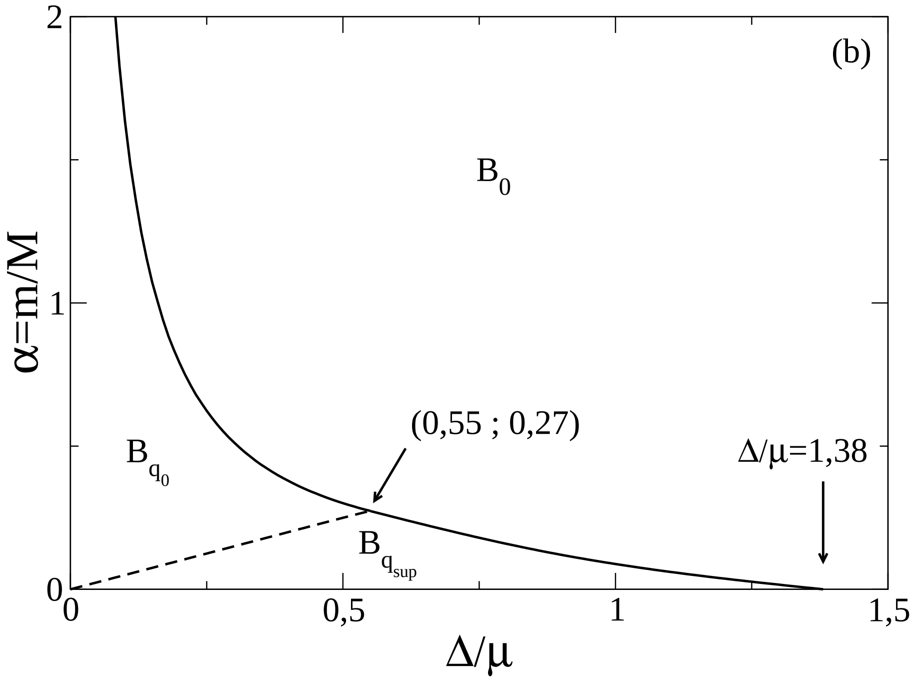
$$0 = \alpha - F_b(q_0) \text{ and } \frac{d}{dq}F_b(q_0) < 0 \text{ with } F_b(q) = -\frac{d}{dq}\frac{\epsilon_b(q)}{q}$$

- **Graphical solution of $\alpha = F_b(q)$ for $\Delta = 0.31$:**

$$v_b(q_0^{\text{inside}}) - c = A_+ - A_-$$

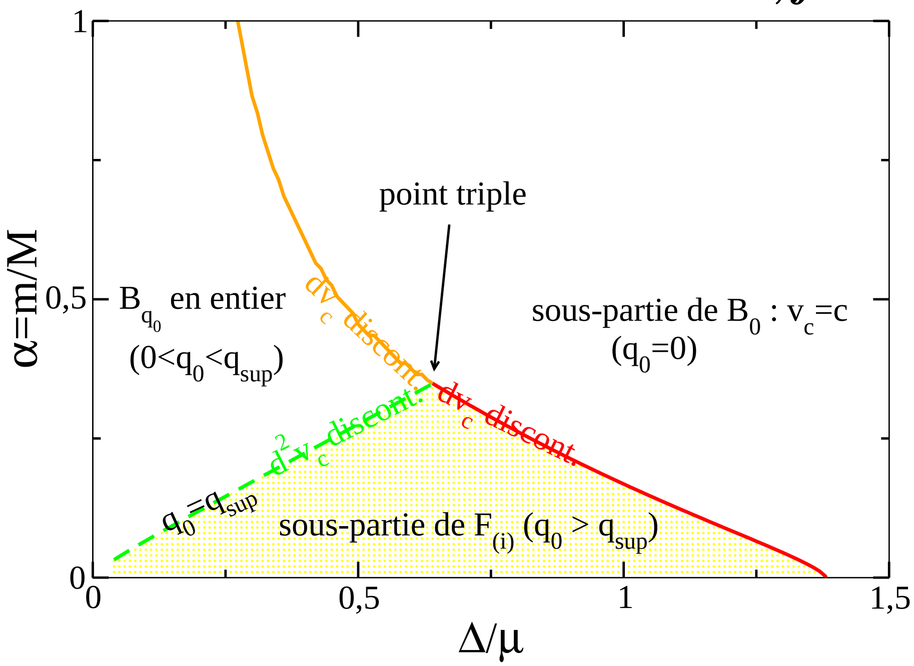


$$\boxed{\begin{array}{l} \alpha^{\max} < \alpha \text{: } q_0 = 0 \\ F_b(q_{\text{sup}}) < \alpha < \alpha^{\max}\text{: } q_0 \in]0, q_{\text{sup}}[\\ \alpha < F_b(q_{\text{sup}})\text{: } q_0 = q_{\text{sup}} \end{array}}$$



- Similarly to fermionic branch: at boundary $B_{q_0} - B_{q_{\mathrm{sup}}}$, leading order discontinuity is the one of $\frac{d^2}{d\alpha^2} v_{c,b}$
- At the other boundaries, leading order discontinuity is the one of $\frac{d}{d\alpha} v_{c,b}$

SYNTHESIS: $v_c = \min(v_{c,f}, v_{c,b})$



Some simple facts coming among others from $2k_{\min} < q_{\rm sup}$:

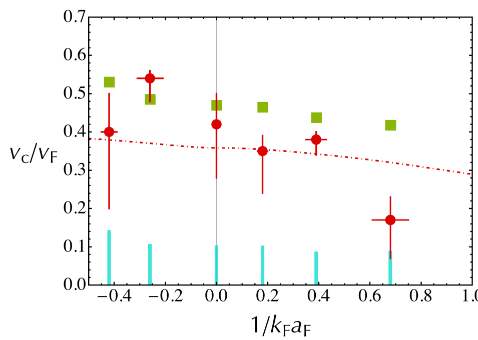
- $\epsilon_f^{\rm eff}(q_{\rm sup}) \leq \epsilon_b(q_{\rm sup})$ so $B_{q_{\rm sup}}$ is masked by $F_{(i)}$
- over its existence domain, $\epsilon_b(q) \leq \epsilon_f^{\rm eff}(q)$ so $F_{(iv)}$ is masked by $B_{q_0} \cup B_0$
- $B_{q_0} - F_{(i)}$ boundary $= B_{q_0} - B_{q_{\rm sup}}$ boundary

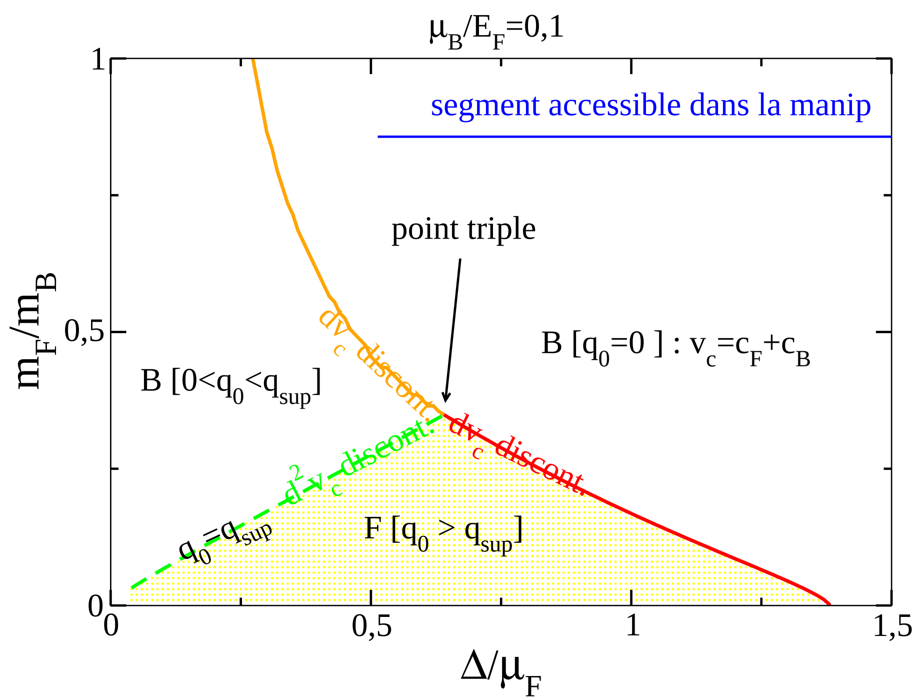
LANDAU FOR THE ENS SYSTEM (Salomon, PRL, 2015)

Experiments at ENS: superfluid cold atom mixtures

- object = small condensate of bosons of velocity $\mathbf{v}$ (^{7}Li). Bogoliubov excitation spectrum $\epsilon_{\mathbf{q}}^{\mathrm{Bog}} + \hbar\mathbf{q}\cdot\mathbf{v}$. Moves in a big gas of spin-1/2 fermions at rest (^{6}Li).
- conservation of energy $\hbar\mathbf{q}\cdot\mathbf{v} = \epsilon_{-\mathbf{q}}^{\mathrm{Bog}} + \epsilon_{\mathbf{q}}$ impossible if

$$v < v_c = \inf_{\mathbf{q}} \frac{\epsilon_{\mathbf{q}}^{\mathrm{Bog}} + \epsilon_{\mathbf{q}}}{\hbar q}$$





5. TEMPORAL COHERENCE: THERMAL BLURRING OF THE CONDENSATE PHASE

"Brouillage thermique d'un gaz cohérent de fermions", Comptes Rendus Physique (in press) [english version arXiv:1502.05644]

DEFINITION OF THE PROBLEM

The considered system:

- A trapped, unpolarized, interacting gas of fermions of spin $1/2$, prepared at thermal equilibrium at $0 < T \ll T_c$
- a condensate of pairs in presence of a weak density of thermal excitations
- the gas is isolated from the environment in its further evolution
- May be realised with cold atoms !

The question we raise:

- what is the temporal pair coherence of the gas ?
- at long times, it is dominated by the condensate coherence

- the condensate coherence time is the width of the function
$$g_1(t) = \langle \hat{a}_0^\dagger(t)\hat{a}_0(0)\rangle$$
where $\hat{a}_0$ annihilates a pair in the condensate mode
$$\hat{a}_0 = \int \mathrm{d}^3r\mathrm{d}^3r'\varphi(\mathbf{r},\mathbf{r}')\hat{\psi}_\downarrow(\mathbf{r})\hat{\psi}_\uparrow(\mathbf{r}')$$

A generalized statistical ensemble:

- the system is in a statistical mixture of many-body eigenstates $|\psi_\lambda\rangle$ with eigenenergies E_λ
- solve the problem for the pure state $|\psi_\lambda\rangle$:
$$g_1^\lambda(t) = \langle \hat{a}_0^\dagger(t)\hat{a}_0(0)\rangle_\lambda$$
- Equivalent to microcanonical ensemble, cf. Eigenstate Thermicity Hypothesis (ETH)
- Finally average over statistical mixture

MODULUS-PHASE REPRESENTATION

$$\hat{a}_0 = \mathrm{e}^{\mathrm{i}\hat{\theta}_0}\hat{N}_0^{1/2}$$

- $\hat{N}_0$ is the number-of-condensed-pairs operator
- $\hat{\theta}_0$ is the condensate phase operator

Neglecting the fluctuations of the modulus:

- For a large system, low relative fluctuations of the number of condensed pairs: $\hat{N}_0 \simeq \bar{N}_0$

$$g_1^\lambda(t) \simeq \bar{N}_0 \mathrm{e}^{\mathrm{i}E_\lambda t/\hbar}\langle \mathrm{e}^{-\mathrm{i}\hat{\theta}_0}\mathrm{e}^{-\mathrm{i}\hat{H}t/\hbar}\mathrm{e}^{\mathrm{i}\hat{\theta}_0}\rangle_\lambda$$

- Introducing

$$\hat{W} \equiv \mathrm{e}^{-\mathrm{i}\hat{\theta}_0}\hat{H}\mathrm{e}^{\mathrm{i}\hat{\theta}_0} - \hat{H} = -\mathrm{i}[\hat{\theta}_0, \hat{H}] + \ldots = O(\hat{N}^0)$$

one obtains

$$g_1^\lambda(t) \simeq \bar{N}_0 e^{iE_\lambda t/\hbar}\langle\psi_\lambda|\mathrm{e}^{-\mathrm{i}(\hat{H}+\hat{W})t/\hbar}|\psi_\lambda\rangle$$

REINTERPRETING THE PROBLEM

$$g_1^{\lambda}(t) \simeq \bar{N}_0 e^{iE_\lambda t/\hbar} \langle \psi_\lambda | \mathrm{e}^{-\mathrm{i}(\hat{H}+\hat{W})t/\hbar} | \psi_\lambda \rangle$$

This is the probability amplitude that the system, being initially in state $|\psi_\lambda\rangle$, is still in state $|\psi_\lambda\rangle$ after an evolution time t in presence of the weak perturbation $\hat{W}$.

Classic problem of a state weakly coupled to a quasi-continuum:

In thermodynamic limit, the perturbation has two effects:

- **energy shift: perturbed energy $E_\lambda + \langle \psi_\lambda | \hat{W} | \psi_\lambda \rangle + O(N^-1)$**
- **decay with a rate given by Fermi golden rule:**

$$\gamma_\lambda = \frac{\pi}{\hbar} \sum_{\mu \neq \lambda} |\langle \psi_\mu | \hat{W} | \psi_\lambda \rangle|^2 \, \delta_\eta(E_\lambda - E_\mu)$$

$$g_1^{\lambda}(t) \simeq \bar{N}_0 \, \mathrm{e}^{-\mathrm{i}t\langle \hat{W} \rangle_\lambda/\hbar} \, \mathrm{e}^{-\gamma_\lambda \mathrm{t}}$$

PHYSICAL INTERPRETATION

$$\hat{W} = \hbar \frac{d\hat{\theta}_0}{dt} + O\left(\frac{1}{N}\right)$$

Coarse grained time derivative of the phase operator:

$$-\frac{\hbar}{2}\overline{\frac{d\hat{\theta}_0}{dt}}^{t} = \mu_0(\hat{N}) + \sum_{s=f,b} \sum_{\alpha} \frac{d\epsilon_{s,\alpha}}{dN} \hat{n}_{s,\alpha}$$

where μ_0 = ground state chemical potential, $\hat{n}_{s,\alpha}$ = quasi-particle occupation number operator in the two branches $s = f, b$. Its expectation value in eigenstate ψ_λ is

$$\langle \frac{d\hat{\theta}_0}{dt} \rangle_\lambda \underset{\text{ETH}}{\simeq} -2\mu_{\text{mc}}(E_\lambda, N_\lambda)/\hbar$$

[adiabatic derivative (at fixed occupation numbers) of energy = microcanonical chemical potential]. This is the second Josephson relation on the order-parameter phase.

- A microscopic derivation using RPA and time-dependent BCS-type variational ansatz in section 3
- At low temperature, where only bosonic branch matters, also predicted by quantum hydrodynamic theory.
- A quantum generalization of the 2nd Josephson relation.

Physical interpretation of γ_λ:

- From a closure relation:

$$\gamma_\lambda = \int_0^{+\infty} \mathrm{d}t \left[\mathrm{Re} \langle \frac{\mathrm{d}\hat{\theta}_0(t)}{\mathrm{d}t} \frac{\mathrm{d}\hat{\theta}_0(0)}{\mathrm{d}t} \rangle_\lambda - \langle \frac{\mathrm{d}\hat{\theta}_0}{\mathrm{d}t} \rangle_\lambda^2 \right] = O(1/N)$$

- This is the phase diffusion coefficient :

$$\gamma_\lambda \underset{\mathrm{ETH}}{=} D(E_\lambda, N_\lambda)$$

- Equation for $\frac{\mathrm{d}\hat{\theta}_0(t)}{\mathrm{d}t}$ plus kinetic equations describing the quasi-particles collisions allow us to calculate γ_λ

TAKING THE STATISTICAL AVERAGE

$$\boxed{g_1^\lambda(t) \simeq \bar{N}_0 \mathrm{e}^{2\mathrm{i}t\mu_{\mathrm{mc}}(E_\lambda,N_\lambda)/\hbar}\mathrm{e}^{-D(E_\lambda,N_\lambda)t}}$$

Average $\mathrm{e}^{2\mathrm{i}t\mu_{\mathrm{mc}}(E_\lambda,N_\lambda)/\hbar}$, linearising μ_{mc} around $(\bar{E},\bar{N})$ and approximate D by its central value: extra Gaussian decay factor

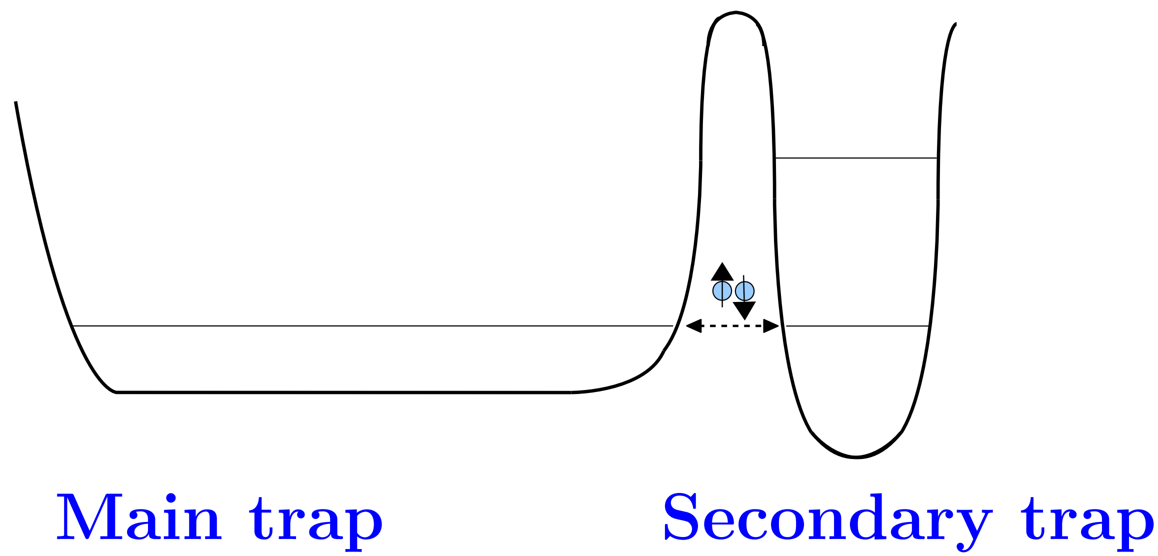
$$g_1(t) \simeq \bar{N}_0 \mathrm{e}^{2\mathrm{i}\mu_{\mathrm{mc}}(\bar{E},\bar{N})t/\hbar}\mathrm{e}^{-\mathrm{t}^2/2\mathrm{t}_{\mathrm{br}}^2}\mathrm{e}^{-D(\bar{E},\bar{N})t}$$

with characteristic time

$$(2t_{\mathrm{br}}/\hbar)^{-2} = \mathrm{Var}\left[N\frac{\partial\mu_{\mathrm{mc}}}{\partial N}(\bar{E},\bar{N}) + E\frac{\partial\mu_{\mathrm{mc}}}{\partial E}(\bar{E},\bar{N})\right]$$

Whenever the two conserved quantities E or N fluctuate, ballistic spreading of the phase distribution.

PROPOSED MEASUREMENT SCHEME



- Ramsey interferometry to measure $g_1(t)$
- Two weak pulses separated by time t: at most one pair transferred to the secondary trap
- Dimerize the pairs for preparation, pulses and detection
- $\langle n_{\mathrm{sec}} \rangle$ oscillates at $\omega = 2(\mu_{\mathrm{main}} - \mu_{\mathrm{sec}})/\hbar$, the contrast is $|g_1(t)/g_1(0)|$

UNITARY FERMI GAS IN CANONICAL ENSEMBLE

- One only needs the equation of state, measured at ENS and MIT, at $T < T_c$:

$$\mu_{\rm mc}(E_{\rm can}(T)) \simeq \mu_{\rm can}(T) \to \partial_E \mu_{\rm mc} \simeq \frac{\partial_T \mu_{\rm can}}{\partial_T E_{\rm can}}$$

$$\mathrm{Var}\, E = k_B T^2 \partial_T \bar{E}$$

- can be estimated by the one of an ideal gas of quasi-particles, keeping the leading order in T for each branch:

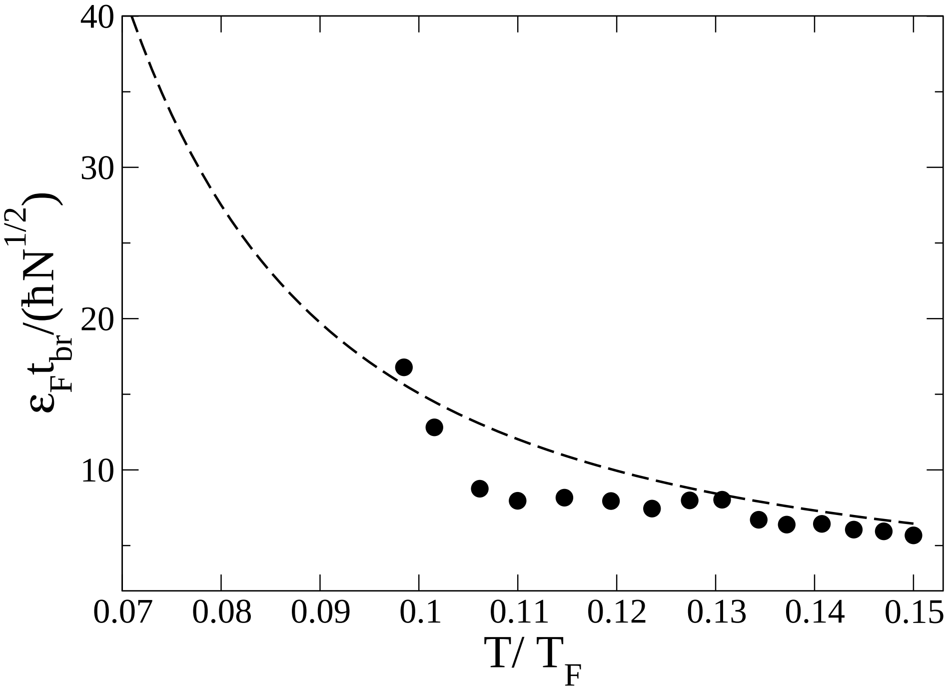
$$\frac{N\hbar^2}{(t_{\rm br}\epsilon_F)^2} \simeq \left(\frac{\theta}{0.46}\right)^5 \frac{(1+2r)^2}{(1+r)}$$

where $\epsilon_F = k_B T_F$ is the Fermi energy, $\theta = T/T_F$ and r is the relative weight of the two branches:

$$r \simeq \left(\frac{0.316}{\theta}\right)^{9/2} \mathrm{e}^{-0.44/\theta}$$

UNITARY FERMI GAS IN CANONICAL ENSEMBLE

Discs: from the equation of state measured at MIT.
Dashed line: approximate formula (ideal gas of quasi-particles)



Typical values: For $T = 0.12T_F \simeq 0.7T_c$, $N = 10^5$, $T_F = 1\mu K$, $t_{\mathrm{br}} = 20\mathrm{ms}$

PHASE DIFFUSION OF UNITARY GAS AT LOW T

- one only keeps the bosonic excitation branch
- the branch is convex at low q:

$$\epsilon_{b,\mathbf{q}} \underset{q\to 0}{=} \hbar c q \left[1 + \frac{\gamma}{8}\left(\frac{\hbar q}{mc}\right)^2 + O(q^4)\right], \quad \gamma_{\mathrm{RPA}} \simeq 0.1$$

- Kinetic equations for the quasiparticle numbers including the Beliaev-Landau decay mechanism
- Diffusion coefficient at low temperature

$$\frac{\hbar N D}{\varepsilon_F} \underset{\theta\to 0}{\sim} C\,\theta^4 \qquad \text{with} \qquad C \propto \gamma^2, C \simeq 0.4$$

- For $T_F = 1\mu K$, increasing the temperature to $T = 0.16 T_F = 0.95 T_c$ and decreasing the atom number to $N = 500$ we find $t_{\mathrm{br}}^{\mathrm{D}} \simeq 15 s$

CONCLUSION OF SECTION 5

- We calculated the intrinsic coherence time of a condensate of paired fermionic atoms at thermal equilibrium.
- Coherence time $\leftrightarrow$ phase dynamics, and $d\hat{\theta}_0/dt \propto$ "chemical potential operator" including pair-breaking and pair-motion excitations.
- As $\hat{\theta}_0(t) \simeq -2\mu_{\rm mc}(E)t/\hbar$, energy fluctuations from one realization to the other $\rightarrow$ Gaussian decay of the coherence $t_{\rm br} \propto N^{1/2}$.
- In the absence of energy fluctuations, the coherence time scales as N due to the diffusive motion of $\hat{\theta}_0$.
- Measurement proposition with cold atoms. We predict $t_{\rm br} \simeq$ 20ms for the canonical ensemble unitary Fermi gas.

6. MAXIMIZING THE INTERACTIONS: THE UNITARY LIMIT

OUTLINE OF SECTION 6

- **What is the unitary gas ?**
- **Separability in hyperspherical coordinates**
- **The Efimov effect**
- **Cluster or virial expansion in the unitary limit**

DEFINITION OF THE UNITARY GAS

- Opposite spin two-body scattering amplitude

$$f_k = -\frac{1}{ik} \quad \forall k$$

- "Maximally" interacting: Unitarity of S matrix imposes $|f_k| \leq 1/k$.
- In real experiments with magnetic Feshbach resonance:

$$-\frac{1}{f_k} = \frac{1}{a} + ik - \frac{1}{2}k^2 r_e + O(k^4 b^3)$$

unitary if "infinite" scattering length a and "zero" ranges:

$$k_{\rm typ}|a| > 100, k_{\rm typ}|r_e| \text{ and } k_{\rm typ} b < \frac{1}{100}$$

imposing $|a| > 10$ microns for $r_e \sim b \sim$ a few nm.

- All these two-body conditions are only necessary.

THE ZERO-RANGE WIGNER-BETHE-PEIERLS MODEL

- **Interactions are replaced by contact conditions.**
- **For $r_{ij} \to 0$ with fixed ij-centroid $\vec{C}_{ij} = (\vec{r}_i + \vec{r}_j)/2$ different from $\vec{r}_k, k \neq i, j$:**

$$\psi(\vec{r}_1, \ldots, \vec{r}_N) = \left(\frac{1}{r_{ij}} - \frac{1}{\mathrm{a}}\right) A_{ij}[\vec{C}_{ij}; (\vec{r}_k)_{k \neq i,j}] + O(r_{ij})$$

- **Elsewhere, non interacting Schrödinger equation**

$$E\psi(\vec{X}) = \left[-\frac{\hbar^2}{2m}\Delta_{\vec{X}} + \frac{1}{2}m\omega^2 X^2\right]\psi(\vec{X})$$

with $\vec{X} = (\vec{r}_1, \ldots, \vec{r}_N)$.

- **Odd exchange symmetry of ψ for same-spin fermion positions.**
- **Unitary gas exists iff Hamiltonian is self-adjoint.**

SCALING INVARIANCE OF CONTACT CONDITIONS

$$\psi(\vec{X}) \underset{r_{ij}\to 0}{=} \frac{1}{r_{ij}} A_{ij}[\vec{C}_{ij}; (\vec{r}_k)_{k\neq i,j}] + O(r_{ij})$$

- Domain of Hamiltonian is scaling invariant: If ψ obeys the contact conditions, so does ψ_λ with

$$\psi_\lambda(\vec{X}) \equiv \frac{1}{\lambda^{3N/2}} \psi(\vec{X}/\lambda)$$

- Simple consequences (also true for the ideal gas):

free space	box (periodic b.c.)	harm. trap
no bound state$^{(*)}$	$PV = 2E/3$ $^{(**)}$	virial $E = 2E_{\rm harm}$ $^{(***)}$

$^{(*)}$ If ψ of eigenenergy E, ψ_λ of eigenenergy E/λ^2. Square integrable eigenfunctions (after center of mass removal) correspond to point-like spectrum, for selfadjoint H. $^{(**)}$ $E(N, V\lambda^3, S) = E(N, V, S)/\lambda^2$, then take derivative in $\lambda = 1$. $^{(***)}$ For eigenstate ψ, mean energy of ψ_λ, $E_\lambda = \frac{\langle H_{\rm Laplacian}\rangle}{\lambda^2} + \langle H_{\rm harm}\rangle\lambda^2$, stationary in $\lambda = 1$.

SEPARABILITY IN HYPERSPHERICAL COORDINATES

- Use Jacobi coordinates to separate center of mass $\vec{C}$
- Hyperspherical coordinates (arbitrary masses m_i):

$$(\vec{r}_1, \ldots, \vec{r}_N) \leftrightarrow (\vec{C}, R, \vec{\Omega}\,)$$

with $3N - 4$ hyperangles $\vec{\Omega}$ and the hyperradius

$$\bar{m}R^2 = \sum_{i=1}^{N} m_i(\vec{r}_i - \vec{C}\,)^2$$

where $\bar{m}$ is the mean mass.

- Hamiltonian is clearly separable:

$$H_{\text{internal}} = -\frac{\hbar^2}{2\bar{m}}\left[\partial_R^2 + \frac{3N-4}{R}\partial_R + \frac{1}{R^2}\Delta_{\vec{\Omega}}\right] + \frac{1}{2}\bar{m}\omega^2 R^2$$

Do the contact conditions preserve separability ?

- For free space $E = 0$, yes, due to scaling invariance:

$$\psi_{E=0} = R^{s-(3N-5)/2}\phi(\vec{\Omega})$$

$E = 0$ Schrödinger's equation implies

$$\Delta_{\vec{\Omega}}\phi(\vec{\Omega}) = -\left[s^2 - \left(\frac{3N-5}{2}\right)^2\right]\phi(\vec{\Omega})$$

with contact conditions. $s^2 \in$ discrete real set.

- For arbitrary E, Ansatz with $E = 0$ hyperrangular part obeys contact conditions $[R^2 = R^2(r_{ij} = 0) + O(r_{ij}^2)]$:

$$\psi = F(R)R^{-(3N-5)/2}\phi(\vec{\Omega})$$

- Schrödinger's equation for a fictitious particle in 2D:

$$EF(R) = -\frac{\hbar^2}{2\bar{m}}\Delta_R^{2D}F(R) + \left[\frac{\hbar^2 s^2}{2\bar{m}R^2} + \frac{1}{2}\bar{m}\omega^2 R^2\right]F(R)$$

SOLUTION OF HYPERRADIAL EQUATION $(N \geq 3)$

$$EF(R) = -\frac{\hbar^2}{2\bar{m}}\Delta_R^{2D} F(R) + \left[\frac{\hbar^2 s^2}{2\bar{m}R^2} + \frac{1}{2}\bar{m}\omega^2 R^2\right] F(R)$$

- Which boundary condition for $F(R)$ in $R = 0$? Wigner-Bethe-Peierls does not say.
- Key point: particular solutions $F(R) \sim R^{\pm s}$ for $R \to 0$.
- Distinguish according to the sign of s^2.

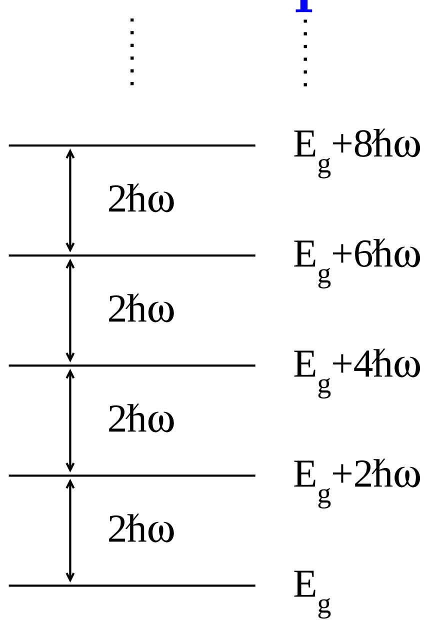
Case $s^2 > 0$

$$F(R) \underset{R\to 0}{\simeq} C_+ R^s + C_- R^{-s}$$

Defining $s > 0$, one discards as usual the divergent solution:

$$F(R) \underset{R\to 0}{\sim} R^s \longrightarrow E_q = E_{\mathrm{CoM}} + (s + 1 + 2q)\hbar\omega, \quad q \in \mathbb{N}$$

then a ladder structure of the spectrum



Case $s^2 < 0$

$$F(R) \underset{R\to 0}{\simeq} C_+ R^s + C_- R^{-s}$$

- To make the Hamiltonian self-adjoint, one is forced to introduce an extra parameter κ (inverse of a length, calculable via microscopic model). For $s = i|s|$:

$$F(R) \underset{R\to 0}{\sim} (\kappa R)^s - (\kappa R)^{-s}$$

- This breaks scaling invariance of the domain. In free space, a geometric spectrum of N-mers:

$$E_n \propto -\frac{\hbar^2 \kappa^2}{\bar{m}} e^{-2\pi n/|s|}, \quad n \in \mathbb{Z}$$

For $N = 3$, this is the Efimov effect:

- Efimov (1971): Solution for three bosons ($1/a = 0$). There exists a single purely imaginary $s_3 \simeq i \times 1.00624$.

- **Efimov (1973): Solution for three arbitrary particles $(1/a = 0)$. Efimov trimers for two fermions (masse m, same spin state) and one impurity (masse m') iff (Petrov, 2003)**

$$\alpha \equiv \frac{m}{m'} > \alpha_c(2;1) \simeq 13.6069$$

ARE THERE EFIMOVIAN TETRAMERS ?

$$E_n^{(4)} \propto -\frac{\hbar^2 \kappa_4^2}{m} e^{-2\pi n/|s_4|} \ ?$$

Negative results for bosons:

- **Amado, Greenwood (1973): "There is No Efimov effect for Four or More Particles". Explanation: Case of bosons, there exist trimers, tetramers decay.**
- **Hammer, Platter (2007), von Stecher, D'Incao, Greene (2009), Deltuva (2010): The four-boson problem (here $1/a = 0$) depends only on κ_3, no κ_4 to add.**
- **Key point: $N = 3$ Efimov effect breaks separability in hyperspherical coordinates for $N = 4$.**

Here, we are dealing with fermions. In short: a 4-body Efimov effect for $3 + 1$ fermions, none for $2 + 2$ fermions.

THE $3+1$ FERMIONIC PROBLEM
(Castin, Mora, Pricoupenko, 2010)

- Three fermions (mass m, same spin state) and one impurity (mass m')
- Our def. of 4-body Efimov effect requires a mass ratio

$$\alpha \equiv \frac{m}{m'} < \alpha_c(2;1) \simeq 13.6069$$

- Calculate $E=0$ solution in momentum space. An integral equation for Fourier transform of A_{ij}:

$$0 = \left[\frac{1+2\alpha}{(1+\alpha)^2}(k_1^2+k_2^2) + \frac{2\alpha}{(1+\alpha)^2}\vec{k}_1\cdot\vec{k}_2\right]^{1/2} D(\vec{k}_1,\vec{k}_2)$$

$$+\int \frac{d^3k_3}{2\pi^2}\frac{D(\vec{k}_1,\vec{k}_3)+D(\vec{k}_3,\vec{k}_2)}{k_1^2+k_2^2+k_3^2+\frac{2\alpha}{1+\alpha}(\vec{k}_1\cdot\vec{k}_2+\vec{k}_1\cdot\vec{k}_3+\vec{k}_2\cdot\vec{k}_3)}$$

- D has to obey fermionic symmetry.

RESULTS

- **Four-body Efimov effect obtained for a single s_4, in channel $l = 1$ with even parity. Corresponding ansatz:**

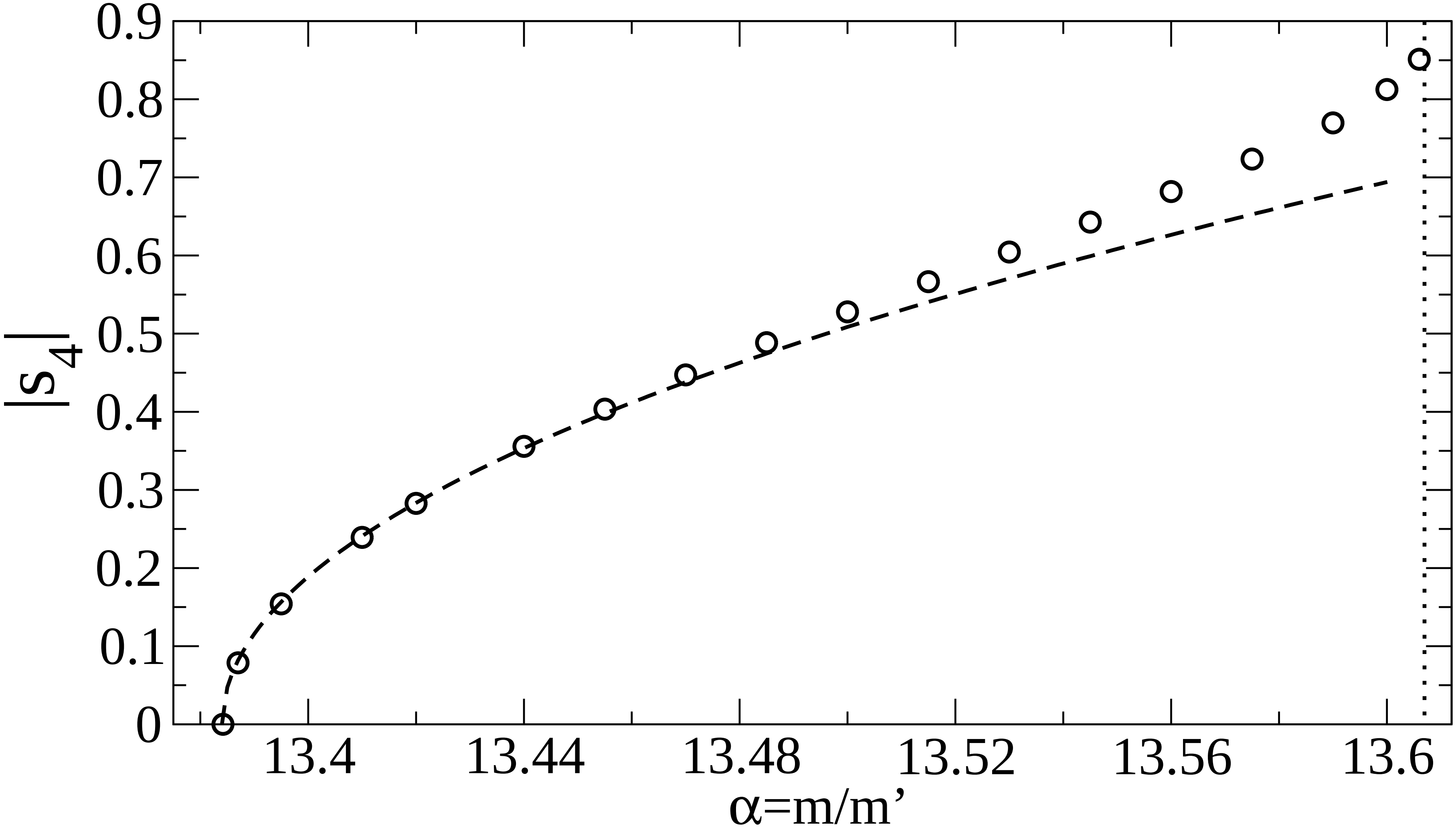
$$D(\vec{k}_1, \vec{k}_2) = \vec{e}_z \cdot \frac{\vec{k}_1 \times \vec{k}_2}{||\vec{k}_1 \times \vec{k}_2||} (k_1^2 + k_2^2)^{-(s_4+7/2)/2} F(k_2/k_1, \theta)$$

in the interval of mass ratio

$$\alpha_c(3;1) \simeq 13.384 < \alpha < \alpha_c(2;1) \simeq 13.607$$

- **In experiments: Use optical lattice to tune effective mass of ^{40}K and ^{3}He* away from $\alpha \simeq 13.25$**

NUMERICAL VALUES OF $s_4 \in i\mathbb{R}$



A CONJECTURE FOR THE FOURTH CLUSTER COEFFICIENTS OF THE UNITARY FERMI GAS

Shimpei Endo, Yvan Castin

REMINDER ON THE CLUSTER EXPANSION

- spatially homogeneous Fermi gas at thermal equilibrium in grand canonical ensemble
- low-density or non-degenerate limit: fugacities $z_\sigma = \exp(\mu_\sigma/k_B T) \to 0$. Series expansion of the total pressure:
$$\frac{P\lambda^3}{k_B T} = 2\sum_{n_1,n_2} b_{n_1,n_2} z_1^{n_1} z_2^{n_2}$$
- Measured at ENS up to order 4 in the unpolarised case $z_1 = z_2$ in the unitary limit $1/a = 0$: difference with ideal gas value
$$\Delta b_4 = 0.096(15)$$
- Theoretically challenging: requires the solution of all possible up to four body problems

Take advantage of scale invariance of the unitary gas:

- The zero-energy free-space solution of Schrödinger's equation is scale invariant with scaling exponent s
- In momentum space, a homogeneous integral equation:

$$M(s)[\tilde{\Phi}_{\text{contact}}] = 0$$

so implicit equation for s:

$$\Lambda(s) \equiv \det M(s) = 0$$

- Once all the possible values of s for few-bodies are known, one knows the energy levels in an isotropic harmonic trap after separation of the center-of-mass:

$$E_q^{\text{rel}} = (2q + s + 1)\hbar\omega, \quad q \in \mathbb{N}$$

so one gets the few-body partition functions $Z^{\text{rel}}_{n_1,n_2}$ and the cluster coefficients B_{n_1,n_2} of the trapped system. The limit $\omega \to 0$ gives access to the b_{n_1,n_2}.

A 3-BODY INSPIRED CONJECTURE FOR b_4

$$\Delta B_{1,1} = \Delta Z_{1,1}^{\mathrm{rel}}$$
$$\Delta B_{2,1} = \Delta Z_{2,1}^{\mathrm{rel}} - Z_1 \Delta B_{1,1}$$

$$I_{N_\uparrow,N_\downarrow} \equiv \int_{\mathbb{R}} \frac{\mathrm{d}S}{2\pi} \frac{\sin(\bar{\omega} S)}{2\sinh\bar{\omega}} \frac{\mathrm{d}}{\mathrm{d}S} \left[\ln \Lambda(\mathrm{i}S)\right]$$

We have shown with F. Werner for bosons, and with Chao Gao and Shimpei Endo for fermions, that $I_{2,1} = \Delta B_{2,1}$ so

$$I_{2,1} = \Delta Z_{2,1}^{\mathrm{rel}} - Z_1 \Delta B_{1,1}$$

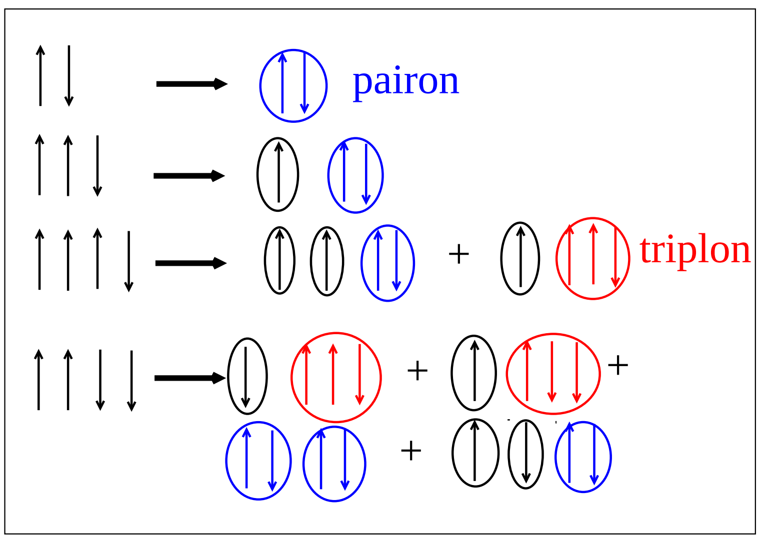
$$I_{2,1} = \Delta Z^{\mathrm{rel}}_{2,1} - Z_1 \Delta B_{1,1}$$

Generalisation:

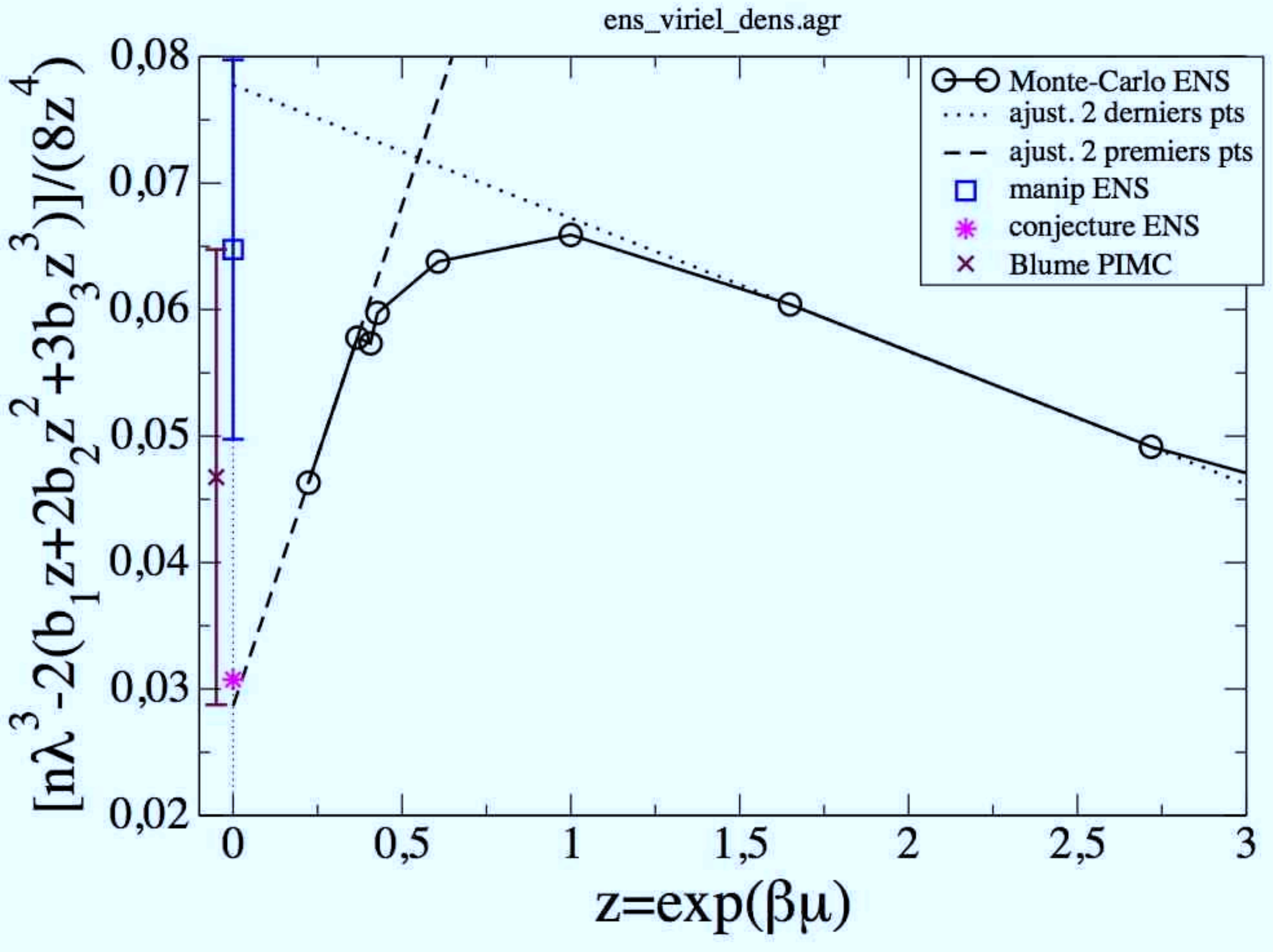
$$I_{3,1} \stackrel{?}{=} \Delta Z^{\mathrm{rel}}_{3,1} - Z_{2,0}\Delta B_{1,1} - Z_1 \Delta B_{2,1}$$

$$I_{2,2} \stackrel{?}{=} \Delta Z^{\mathrm{rel}}_{2,2} - Z_1 \Delta B_{2,1} - Z_1 \Delta B_{1,2}$$
$$- \Delta Z^{\mathrm{rel}}_{2\,\mathrm{pairons}} - Z_1 (Z_1 - Z^{\mathrm{rel,ideal}}_{1\,\mathrm{pairon}}) \Delta B_{1,1}$$

Decoupled Asymptotic Objects
(at large quantum numbers)



Gives $\Delta b_4 \simeq 0.06 \neq 0.096(15)$.



ens_viriel_dens.agr
Monte-Carlo ENS
ajust. 2 derniers pts
ajust. 2 premiers pts
manip ENS
conjecture ENS
Blume PIMC
$[n\lambda^3-2(b_1z+2b_2z^2+3b_3z^3)]/(8z^4)$
$z=\exp(\beta\mu)$
0,02
0,03
0,04
0,05
0,06
0,07
0,08
0
0,5
1
1,5
2
2,5
3